9 Sistemi linearnih jednačina

Zadatak 9.1 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r} 4x-3y-2z=1 \\ -2x+y+3z=-1 \\ x+y-z=2 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 4 & -3 & -2 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \\ A_p &= \left[\begin{array}{rrr|r} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{rrr|r} 4 & -3 & -2 & 1 \\ -2 & 1 & 3 & -1 \\ 1 & 1 & -1 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ -2 & 1 & 3 & -1 \\ 4 & -3 & -2 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 7 & -2 & 7 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ 2\cdot V_1+V_2 \\ 4\cdot V_1-V_3 \end{array} \\ &\sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 13 & 0 \end{array}\right] \begin{array}{l} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ 7\cdot V_2-3\cdot V_3 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A)=3$ i $rang\,(A_p)=3$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{array}{r} x+y-z=2 \\ 3y+z=3 \\ 13z=0 \end{array}\right\}.$$

Iz treće jednačine vidimo da je $z=0$ i ako to uvrstimo u drugu jednačinu dobićemo

$$\begin{aligned} 3y+0 &= 3 \\ 3y &= 3 \Longrightarrow y=1. \end{aligned}$$

Uvrštavajući $z=0$ i $y=1$ u prvu jednačinu, dobijamo

$$\begin{aligned} x+1-0 &= 2 \\ x+1 &= 2 \Longrightarrow x=2. \end{aligned}$$

Rješenje sistema je

$$(x,y,z)=(1,1,0)\,.$$

Zadatak 9.2 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r} 3x-2y+z=2 \\ -x+y-3z=-3 \\ 2x-y-2z=-1 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned} A &= \begin{bmatrix} 3 & -2 & 1 \\ -1 & 1 & -3 \\ 2 & -1 & -2 \end{bmatrix} \\ A_p &= \left[\begin{array}{rrr|r} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{array}\right]. \end{aligned}$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{rrr|r} 3 & -2 & 1 & 2 \\ -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} -1 & 1 & -3 & -3 \\ 2 & -1 & -2 & -1 \\ 3 & -2 & 1 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 1 & -8 & 7 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ 2\cdot V_1+V_2 \\ 3\cdot V_1+V_3 \end{array} \\ &\sim \left[\begin{array}{rrr|r} -1 & 1 & -3 & -3 \\ 0 & 1 & -8 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] . \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ V_2-V_3 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A)=2$ i $rang\,(A_p)=2$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{array}{r} -x+y-3z=-3 \\ y-8z=-7 \end{array}\right\}.$$

Ovaj sistem ima tri nepoznate a dvije jednačine. Zato jednu nepoznatu biramo proizvoljno.

Neka je, recimo, $z = \alpha$ gdje je $\alpha \in \mathbb{R}$.

Ako $z = \alpha$ uvrstimo u drugu jednačinu, dobijamo

$$y - 8\alpha = -7 \implies y = -7 + 8\alpha.$$

Ako $y = -7 + 8\alpha$ i $z = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} -x - 7 + 8\alpha - 3\alpha &= -3 \\ -x - 7 + 5\alpha &= -3 \\ -x &= 4 - 5\alpha \implies x = -4 + 5\alpha \end{aligned}$$

Rješenje sistema je

$$(x, y, z) = (-4 + 5\alpha, -7 + 8\alpha, \alpha), \qquad \alpha \in \mathbb{R}.$$

Zadatak 9.3 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{aligned} 2x - y + 3z + t = 4 \\ x + y + z + t = 3 \\ -x + 2y - 3z - t = -4 \\ x + y + 2z - 2t = 3 \end{aligned}\right\}.$$

Rješenje:

Matrica koeficijenata i proširena matrica ovog sistema je

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 2 & -3 & -1 \\ 1 & 1 & 2 & -2 \end{bmatrix}$$

$$A_p = \left[\begin{array}{cccc|c} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{array}\right]$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{cccc|c} 2 & -1 & 3 & 1 & 4 \\ 1 & 1 & 1 & 1 & 3 \\ -1 & 2 & -3 & -1 & -4 \\ 1 & 1 & 2 & -2 & 2 \end{array}\right] \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 & 2 \\ -1 & 2 & -3 & -1 & -4 \\ 2 & -1 & 3 & 1 & 4 \end{array}\right] \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & -1 & 3 & 1 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_1 - V_2 \\ V_1 + V_3 \\ 2 \cdot V_1 - V_4 \end{array} \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 3 & -1 & 1 & 2 \\ 0 & 0 & -1 & 3 & 1 \end{array}\right] \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & -1 & 3 & 1 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ V_1 + V_3 \\ V_4 \textit{ prepisana} \end{array} \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 3 & -2 & 0 & -1 \\ 0 & 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & -4 & -4 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ V_3 \textit{ prepisana} \\ V_3 - V_4 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 4$ i $rang\,(A_p) = 4$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{aligned} x + y + z + t = 3 \\ 3y - 2z = -1 \\ -z - t = -3 \\ -4t = -4 \end{aligned}\right\}.$$

Iz četvrte jednačine vidimo da je $t = 1$.
Ako $t = 1$ uvrstimo u treću jednačinu dobićemo

$$\begin{aligned} -z - 1 &= -3 \\ -z &= -2 \implies z = 2. \end{aligned}$$

Ako $z = 2$ i $t = 1$ uvrstimo u drugu jednačinu, dobićemo

$$\begin{aligned} 3y - 4 &= -1 \\ 3y &= 3 \implies y = 1. \end{aligned}$$

Ako $y = 1$, $z = 2$ i $t = 1$ uvrstimo u prvu jednačinu, dobićemo

$$x + 1 + 2 + 1 = 3 \implies x = -1.$$

Rješenje sistema je

$$(x, y, z, t) = (-1, 1, 2, 1).$$

Zadatak 9.4 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{aligned} 2x + 3y - 4z + t = 2 \\ x + y - z + 3t = 4 \\ -x + 3y - 2z + 2t = 2 \\ 3x + 4y - 5z + 4t = 6 \end{aligned}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$A = \begin{bmatrix} 2 & 3 & -4 & 1 \\ 1 & 1 & -1 & 3 \\ -1 & 3 & -2 & 2 \\ 3 & 4 & -5 & 4 \end{bmatrix}$$

$$A_p = \left[\begin{array}{cccc|c} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{array}\right].$$

Odredimo rang proširene matrice:

$$\begin{aligned} A_p &= \left[\begin{array}{cccc|c} 2 & 3 & -4 & 1 & 2 \\ 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{array}\right] \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 4 \\ -1 & 3 & -2 & 2 & 2 \\ 2 & 3 & -4 & 1 & 2 \\ 3 & 4 & -5 & 4 & 6 \end{array}\right] \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \\ 0 & -1 & 2 & 5 & 6 \end{array}\right] \begin{array}{r} V_1 \ \textit{prepisana} \\ V_1 + V_2 \\ 2V_1 - V_3 \\ 3V_1 - V_4 \end{array} \\ &\sim \left[\begin{array}{cccc|c} 1 & 1 & -1 & 3 & 4 \\ 0 & 4 & -3 & 5 & 6 \\ 0 & 0 & 5 & 25 & 30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \ \textit{prepisana} \\ V_2 \ \textit{prepisana} \\ V_2 + 4V_3 \\ V_3 - V_4 \end{array} \end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 3$ i $rang\,(A_p) = 3$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{aligned} x + y - z + 3t = 4 \\ 4y - 3z + 5t = 6 \\ 5z + 25t = 30 \end{aligned}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t = \alpha \ (\alpha \in \mathbb{R})$.
Iz treće jednačine imamo

$$\begin{aligned} 5z + 25\alpha &= 30 \\ z + 5\alpha &= 6 \implies z = 6 - 5\alpha. \end{aligned}$$

Ako $z = 6 - 5\alpha$ i $t = \alpha$ uvrstimo u drugu jednačinu, dobijamo

$$\begin{aligned} 4y - 3\,(6 - 5\alpha) + 5\alpha &= 6 \\ 4y - 18 + 15\alpha + 5\alpha &= 6 \\ 4y &= 24 - 20\alpha \implies y = 6 - 5\alpha. \end{aligned}$$

Ako $y = 6 - 5\alpha, z = 6 - 5\alpha$ i $t = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} x + 6 - 5\alpha - 6 + 5\alpha + 3\alpha &= 4 \\ x + 3\alpha &= 4 \implies x = 4 - 3\alpha. \end{aligned}$$

Rješenje sistema je

$$(x,y,z,t)=(4-3\alpha,6-5\alpha,6-5\alpha,\alpha)\qquad(\alpha\in\mathbb{R}).$$

Zadatak 9.5 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r}3x-2y+2z=3\\x+y-z=1\\4x-y+z=4\\2x-3y+3z=2\end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$A=\begin{bmatrix}3&-2&2\\1&1&-1\\4&-1&1\\2&-3&3\end{bmatrix}$$
$$A_p=\left[\begin{array}{rrr|r}3&-2&2&3\\1&1&-1&1\\4&-1&1&4\\2&-3&3&2\end{array}\right].$$

Odredimo rang proširene matrice:

$$\begin{aligned}A_p&=\left[\begin{array}{rrr|r}3&-2&2&3\\1&1&-1&1\\4&-1&1&4\\2&-3&3&2\end{array}\right]\\
&\sim\left[\begin{array}{rrr|r}1&1&-1&1\\3&-2&2&3\\4&-1&1&4\\2&-3&3&2\end{array}\right]\\
&\sim\left[\begin{array}{rrr|r}1&1&-1&1\\0&-5&5&0\\0&-5&5&0\\0&-5&5&0\end{array}\right]\begin{array}{r}V_1\ prepisana\\V_2-3V_1\\V_3-4V_1\\V_4-2V_1\end{array}\\
&\sim\left[\begin{array}{rrr|r}1&1&-1&1\\0&-5&5&0\\0&0&0&0\\0&0&0&0\end{array}\right]\begin{array}{r}V_1\ prepisana\\V_2\ prepisana\\V_3-V_2\\V_4-V_2\end{array}\end{aligned}$$

Odavdje vidimo da je $rang\,(A)=2$ i $rang\,(A_p)=2$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{array}{r}x+y-z=1\\-5y+5z=0\end{array}\right\}.$$

Ovaj sistem ima tri nepoznate a dvije jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $z=\alpha\ (\alpha\in\mathbb{R})$.
Iz druge jednačine imamo

$$\begin{aligned}-5y+5\alpha&=0\\-5y&=-5\alpha\Longrightarrow y=\alpha.\end{aligned}$$

Ako $y=\alpha$ i $z=\alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned}x+y-z&=1\\x+\alpha-\alpha&=1\Longrightarrow x=1.\end{aligned}$$

Rješenje sistema je

$$(x,y,z)=(1,\alpha,\alpha)\qquad(\alpha\in\mathbb{R}).$$

Zadatak 9.6 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r}3x-2y+z+t=3\\2x+y-2z-t=0\\x+2y-z+t=3\\3x+3y-3z=3\end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$A=\begin{bmatrix}3&-2&1&1\\2&1&-2&-1\\1&2&-1&1\\3&3&-3&0\end{bmatrix}$$
$$A_p=\left[\begin{array}{rrrr|r}3&-2&1&1&3\\2&1&-2&-1&0\\1&2&-1&1&3\\3&3&-3&0&3\end{array}\right].$$

Odredimo rang proširene matrice:

$$A_p = \left[\begin{array}{cccc|c} 3 & -2 & 1 & 1 & 3 \\ 2 & 1 & -2 & -1 & 0 \\ 1 & 2 & -1 & 1 & 3 \\ 3 & 3 & -3 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 2 & 1 & -2 & -1 & 0 \\ 3 & -2 & 1 & 1 & 3 \\ 3 & 3 & -3 & 0 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 0 & -3 & -6 \\ 0 & -8 & 4 & -2 & -6 \\ 0 & -3 & 0 & -3 & -6 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 - 2V_1 \\ V_3 - 3V_1 \\ V_4 - 3V_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 0 & -3 & -6 \\ 0 & 0 & -12 & -18 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \textit{ prepisana} \\ V_2 \textit{ prepisana} \\ 8V_2 - 3V_3 \\ V_2 - V_4 \end{array}$$

Odavdje vidimo da je $rang\,(A) = 3$ i $rang\,(A_p) = 3$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{array}{r} x + 2y - z + t = 3 \\ -3y - 3t = -6 \\ -12z - 18t = -30 \end{array}\right\} \iff \left.\begin{array}{r} x + 2y - z + t = 3 \\ y + t = 2 \\ 2z + 3t = 5 \end{array}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t = \alpha\ (\alpha \in \mathbb{R})$.
Iz treće jednačine imamo

$$\begin{aligned} 2z + 3\alpha &= 5 \\ 2z &= 5 - 3\alpha \implies z = \frac{5 - 3\alpha}{2}. \end{aligned}$$

Ako $z = \frac{5-3\alpha}{2}$ i $t = \alpha$ uvrstimo u drugu jednačinu, dobijamo

$$y + \alpha = 2 \implies y = 2 - \alpha.$$

Ako $y = 2 - \alpha$, $z = \frac{5-3\alpha}{2}$ i $t = \alpha$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned} x + 2\,(2 - \alpha) - \frac{5 - 3\alpha}{2} + \alpha &= 3 \\ x + 4 - 2\alpha - \frac{5 - 3\alpha}{2} + \alpha &= 3 \\ x + \frac{8 - 4\alpha - 5 + 3\alpha + 2\alpha}{2} &= 3 \\ x + \frac{3 + \alpha}{2} &= 3 \\ x &= 3 - \frac{3 + \alpha}{2} \implies x = \frac{3 - \alpha}{2}. \end{aligned}$$

Rješenje sistema je

$$(x, y, z, t) = \left(\frac{3 - \alpha}{2}, 2 - \alpha, \frac{5 - 3\alpha}{2}, \alpha\right) \qquad (\alpha \in \mathbb{R}).$$

Zadatak 9.7 *Ispitati saglasnost i riješiti sistem*

$$\left.\begin{array}{r} 2x + 2y - z + 3t = 6 \\ 3x + y - 3z + t = 2 \\ 5x + 3y - 4z + 4t = 8 \\ x - y - 2z - 2t = -4 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$A = \left[\begin{array}{cccc} 2 & 2 & -1 & 3 \\ 3 & 1 & -3 & 1 \\ 5 & 3 & -4 & 4 \\ 1 & -1 & -2 & -2 \end{array}\right]$$

$$A_p = \left[\begin{array}{cccc|c} 2 & 2 & -1 & 3 & 6 \\ 3 & 1 & -3 & 1 & 2 \\ 5 & 3 & -4 & 4 & 8 \\ 1 & -1 & -2 & -2 & 4 \end{array}\right].$$

Odredimo rang proširene matrice:

$$\begin{aligned}
A_p &= \left[\begin{array}{rrrr|r} 2 & 2 & -1 & 3 & 6 \\ 3 & 1 & -3 & 1 & 2 \\ 5 & 3 & -4 & 4 & 8 \\ 1 & -1 & -2 & -2 & -4 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & -1 & -2 & -2 & -4 \\ 2 & 2 & -1 & 3 & 6 \\ 3 & 1 & -3 & 1 & 2 \\ 5 & 3 & -4 & 4 & 8 \end{array}\right] \\
&\sim \left[\begin{array}{rrrr|r} 1 & -1 & -2 & -2 & -4 \\ 0 & -4 & -3 & -7 & -14 \\ 0 & -4 & -3 & -7 & -14 \\ 0 & -8 & -6 & -14 & -28 \end{array}\right] \begin{array}{r} V_1 \ \textit{prepisana} \\ 2V_1 - V_2 \\ 3V_1 - V_3 \\ 5V_1 - 3V_4 \end{array} \\
&\sim \left[\begin{array}{rrrr|r} 1 & -1 & -2 & -2 & -4 \\ 0 & -4 & -3 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \ \textit{prepisana} \\ 2V_1 - V_2 \\ V_2 - V_3 \\ 2V_2 - V_4 \end{array}
\end{aligned}$$

Odavdje vidimo da je $rang\,(A) = 2$ i $rang\,(A_p) = 2$ pa je sistem saglasan.
Formirajmo novi sistem

$$\left.\begin{array}{r} x - y - 2z - 2t = -4 \\ -4y - 3z - 7t = -14 \end{array}\right\}.$$

Ovaj sistem ima četiri nepoznate a dvije jednačine.
Znači, dvije nepoznate uzimamo proizvoljno.
Neka je, npr. $z = \alpha, t = \beta \ (\alpha, \beta \in \mathbb{R})$.
Iz druge jednačine imamo

$$\begin{aligned}
-4y - 3\alpha - 7\beta &= -14 \\
-4y &= -14 + 3\alpha + 7\beta \Longrightarrow y = \frac{14 - 3\alpha - 7\beta}{4}.
\end{aligned}$$

Ako $y = \frac{14-3\alpha-7\beta}{4}, z = \alpha, t = \beta$ uvrstimo u prvu jednačinu, dobijamo

$$\begin{aligned}
x - \frac{14 - 3\alpha - 7\beta}{4} - 2\alpha - 2\beta &= -4 \\
x + \frac{-14 + 3\alpha + 7\beta - 8\alpha - 8\beta}{4} &= -4 \\
x + \frac{-14 - 5\alpha - \beta}{4} &= -4 \\
x &= -4 - \frac{-14 - 5\alpha - \beta}{4} \\
x &= \frac{-16 + 14 + 5\alpha + \beta}{4} \Longrightarrow x = \frac{-2 + 5\alpha + \beta}{4}
\end{aligned}$$

Rješenje sistema je

$$(x, y, z, t) = \left(\frac{-2 + 5\alpha + \beta}{4}, \frac{14 - 3\alpha - 7\beta}{4}, \alpha, \beta\right) \qquad (\alpha, \beta \in \mathbb{R}).$$

Zadatak 9.8 *Ispitati saglasnost sistema i u slučaju saglasnosti riješiti sistem matričnom metodom*

$$\left.\begin{array}{r} 2x - y + z - 3t = -1 \\ x - y + 3z + t = 4 \\ -2x - 3y + z + t = -3 \\ 3x - 2y + 4z - 2t = 3 \end{array}\right\}.$$

Rješenje:
Matrica koeficijenata i proširena matrica ovog sistema je

$$\begin{aligned}
A &= \left[\begin{array}{rrrr} 2 & -1 & 1 & -3 \\ 1 & -1 & 3 & 1 \\ -2 & -3 & 1 & 1 \\ 3 & -2 & 4 & -2 \end{array}\right] \\
A_p &= \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right].
\end{aligned}$$

Odredimo rang proširene matrice:

$$A_p = \left[\begin{array}{cccc|c} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & -3 & -1 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & -5 & 7 & 3 & 5 \\ 0 & -1 & 5 & 5 & 9 \end{array}\right] \begin{array}{r} V_1 \; prepisana \\ 2V_1 - V_2 \\ 2V_1 + V_3 \\ 3V_1 - V_4 \end{array}$$
$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & 0 & 18 & 22 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \; prepisana \\ V_2 \; prepisana \\ 5V_2 - V_3 \\ V_2 - V_3 \end{array}$$

Odavdje vidimo da je $rang\,(A) = 3$ i $rang\,(A_p) = 3$ pa je sistem saglasan.

Formirajmo novi sistem

$$\left.\begin{array}{r} x - y + 3z + t = 4 \\ -y + 5z + 5t = 9 \\ 18z + 22t = 40 \end{array}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t = \alpha \ (\alpha \in \mathbb{R})$.
Sada sistem ima oblik

$$\left.\begin{array}{r} x - y + 3z = 4 - \alpha \\ -y + 5z = 9 - 5\alpha \\ 9z = 20 - 11\alpha \end{array}\right\}.$$

Ovaj sistem je ekvivalentan matričnoj jednačini

$$AX = B$$

gdje su

$$A = \left[\begin{array}{ccc} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 9 \end{array}\right], \quad X = \left[\begin{array}{c} x \\ y \\ z \end{array}\right], \quad B = \left[\begin{array}{c} 4 - \alpha \\ 9 - 5\alpha \\ 20 - 11\alpha \end{array}\right].$$

Rješenje matrične jednačine je matrica $X = A^{-1}B$.
Izračunajmo matricu A^{-1} na osnovu formule

$$A^{-1} = \frac{1}{\det A} adj A.$$

Determinanta matrice A je

$$\det A = \left|\begin{array}{ccc} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 9 \end{array}\right| = -9$$

Kofaktori matrice A su

$$A_{11} = \left|\begin{array}{cc} -1 & 5 \\ 0 & 9 \end{array}\right| = -9 \quad A_{12} = -\left|\begin{array}{cc} 0 & 5 \\ 0 & 9 \end{array}\right| = 0 \quad A_{13} = \left|\begin{array}{cc} 0 & -1 \\ 0 & 0 \end{array}\right| = 0$$
$$A_{21} = -\left|\begin{array}{cc} -1 & 3 \\ 0 & 9 \end{array}\right| = 9 \quad A_{22} = \left|\begin{array}{cc} 1 & 3 \\ 0 & 9 \end{array}\right| = 9 \quad A_{23} = -\left|\begin{array}{cc} 1 & -1 \\ 0 & 0 \end{array}\right| = 0$$
$$A_{31} = \left|\begin{array}{cc} -1 & 3 \\ -1 & 5 \end{array}\right| = -2 \quad A_{32} = -\left|\begin{array}{cc} 1 & 3 \\ 0 & 5 \end{array}\right| = -5 \quad A_{33} = \left|\begin{array}{cc} 1 & -1 \\ 0 & -1 \end{array}\right| = -1$$

pa je adjungovana matrica

$$adj A = \left[\begin{array}{ccc} -9 & 9 & -2 \\ 0 & 9 & -5 \\ 0 & 0 & -1 \end{array}\right].$$

Inverzna matrica matrice A je

$$A^{-1} = \frac{1}{-9}\left[\begin{array}{ccc} -9 & 9 & -2 \\ 0 & 9 & -5 \\ 0 & 0 & -1 \end{array}\right] = \left[\begin{array}{ccc} 1 & -1 & \frac{2}{9} \\ 0 & -1 & \frac{5}{9} \\ 0 & 0 & \frac{1}{9} \end{array}\right].$$

Sada je

$$X = A^{-1}B = \left[\begin{array}{ccc} 1 & -1 & \frac{2}{9} \\ 0 & -1 & \frac{5}{9} \\ 0 & 0 & \frac{1}{9} \end{array}\right] \cdot \left[\begin{array}{c} 4 - \alpha \\ 9 - 5\alpha \\ 20 - 11\alpha \end{array}\right] = \left[\begin{array}{c} \frac{-5+14\alpha}{9} \\ \frac{19-10\alpha}{9} \\ \frac{20-11\alpha}{9} \end{array}\right]$$

pa je

$$\begin{aligned} x &= \frac{-5 + 14\alpha}{9} \\ y &= \frac{19 - 10\alpha}{9} \\ z &= \frac{20 - 11\alpha}{9}, \qquad \alpha \in \mathbb{R} \end{aligned}$$

rješenje sistema.

Zadatak 9.9 *Ispitati saglasnost sistema i u slučaju saglasnosti riješiti sistem metodom determinanti*

$$\left.\begin{array}{r} 2x-y+z-3t=-1 \\ x-y+3z+t=4 \\ -2x-3y+z+t=-3 \\ 3x-2y+4z-2t=3 \end{array}\right\}.$$

Rješenje:

Matrica koeficijenata i proširena matrica ovog sistema je

$$A = \begin{bmatrix} 2 & -1 & 1 & -3 \\ 1 & -1 & 3 & 1 \\ -2 & -3 & 1 & 1 \\ 3 & -2 & 4 & -2 \end{bmatrix}$$

$$A_p = \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right].$$

Odredimo rang proširene matrice:

$$A_p = \left[\begin{array}{rrrr|r} 2 & -1 & 1 & -3 & -1 \\ 1 & -1 & 3 & 1 & 4 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 2 & -1 & 1 & -3 & -1 \\ -2 & -3 & 1 & 1 & -3 \\ 3 & -2 & 4 & -2 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & -5 & 7 & 3 & 5 \\ 0 & -1 & 5 & 5 & 9 \end{array}\right] \begin{array}{r} V_1 \ \textit{prepisana} \\ 2V_1-V_2 \\ 2V_1+V_3 \\ 3V_1-V_4 \end{array}$$

$$\sim \left[\begin{array}{rrrr|r} 1 & -1 & 3 & 1 & 4 \\ 0 & -1 & 5 & 5 & 9 \\ 0 & 0 & 18 & 22 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{r} V_1 \ \textit{prepisana} \\ V_2 \ \textit{prepisana} \\ 5V_2-V_3 \\ V_2-V_3 \end{array}$$

Odavdje vidimo da je $rang\,(A)=3$ i $rang\,(A_p)=3$ pa je sistem saglasan.

Formirajmo novi sistem

$$\left.\begin{array}{r} x-y+3z+t=4 \\ -y+5z+5t=9 \\ 18z+22t=40 \end{array}\right\}.$$

Ovaj sistem ima četiri nepoznate a tri jednačine.
Znači, jednu nepoznatu uzimamo proizvoljno.
Neka je, npr. $t=\alpha\ (\alpha\in\mathbb{R})$.
Sada sistem ima oblik

$$\left.\begin{array}{r} x-y+3z=4-\alpha \\ -y+5z=9-5\alpha \\ 9z=20-11\alpha \end{array}\right\}.$$

Izračunajmo determinante D, D_x, D_y, D_z :

$$D = \begin{vmatrix} 1 & -1 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 9 \end{vmatrix} = -9$$

$$D_x = \begin{vmatrix} 4-\alpha & -1 & 3 \\ 9-5\alpha & -1 & 5 \\ 20-11\alpha & 0 & 9 \end{vmatrix} = 5-14\alpha$$

$$D_y = \begin{vmatrix} 1 & 4-\alpha & 3 \\ 0 & 9-5\alpha & 5 \\ 0 & 20-11\alpha & 9 \end{vmatrix} = -19+10\alpha$$

$$D_z = \begin{vmatrix} 1 & -1 & 4-\alpha \\ 0 & -1 & 9-5\alpha \\ 0 & 0 & 20-11\alpha \end{vmatrix} = -20+11\alpha$$

pa je

$$x = \frac{D_x}{D} = \frac{-5+14\alpha}{9}$$
$$y = \frac{D_y}{D} = \frac{19-10\alpha}{9}$$
$$z = \frac{D_z}{D} = \frac{20-11\alpha}{9}, \qquad \alpha\in\mathbb{R}$$

rješenje sistema.

Zadatak 9.10 *U zavisnosti od realnog parametra λ diskutovati rješenja sistema*

$$\left.\begin{array}{r} \lambda x+y+z=1 \\ x+\lambda y+z=\lambda \\ x+y+\lambda z=\lambda^2 \end{array}\right\}.$$

Rješenje:

Izračunajmo determinante D, D_x, D_y, D_z :

$$\begin{aligned} D &= \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = \left|\begin{array}{ccc|cc} \lambda & 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 & \lambda \\ 1 & 1 & \lambda & 1 & 1 \end{array}\right| = \\ &= \left(\lambda^3+1+1\right)-(\lambda+\lambda+\lambda)=\lambda^3-3\lambda+2= \\ &= \lambda^3-\lambda-2\lambda+2=\lambda\left(\lambda^2-1\right)-2(\lambda-1)= \\ &= \lambda(\lambda-1)(\lambda+1)-2(\lambda-1)= \\ &= (\lambda-1)[\lambda(\lambda+1)-2]=(\lambda-1)\left[\lambda^2+\lambda-2\right]= \\ &= (\lambda-1)\left[\lambda^2+2\lambda-\lambda-2\right]=(\lambda-1)[\lambda(\lambda+2)-(\lambda+2)]= \\ &= (\lambda-1)(\lambda-1)(\lambda+2)=(\lambda-1)^2(\lambda+2). \\ D_x &= \left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ \lambda & \lambda & 1 & \lambda & \lambda \\ \lambda^2 & 1 & \lambda & \lambda^2 & 1 \end{array}\right| = \\ &= \left(\lambda^2+\lambda^2+\lambda\right)-\left(\lambda^3+1+\lambda^2\right)= \\ &= -\lambda^3+\lambda^2+\lambda-1=-\left(\lambda^3-\lambda^2\right)+(\lambda-1)= \\ &= -\lambda^2(\lambda-1)+(\lambda-1)=(\lambda-1)\left(1-\lambda^2\right)= \\ &= (\lambda-1)(1-\lambda)(1+\lambda)=-(\lambda-1)(\lambda-1)(\lambda+1)= \\ &= -(\lambda-1)^2(\lambda+1). \\ D_y &= \left|\begin{array}{ccc|cc} \lambda & 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 & \lambda \\ 1 & \lambda^2 & \lambda & 1 & \lambda^2 \end{array}\right| = \\ &= \left(\lambda^3+1+\lambda^2\right)-\left(\lambda+\lambda^3+\lambda\right)= \\ &= \lambda^2-2\lambda+1=(\lambda-1)^2. \end{aligned}$$

$$\begin{aligned} D_z &= \left|\begin{array}{ccc|cc} \lambda & 1 & 1 & \lambda & 1 \\ 1 & \lambda & \lambda & 1 & \lambda \\ 1 & 1 & \lambda^2 & 1 & 1 \end{array}\right| = \\ &= \left(\lambda^4+\lambda+1\right)-\left(\lambda+\lambda^2+\lambda^2\right)= \\ &= \lambda^4-2\lambda^2+1=\left(\lambda^2-1\right)^2= \\ &= ((\lambda-1)(\lambda+1))^2=(\lambda-1)^2(\lambda+1)^2. \end{aligned}$$

Dakle,

$$\begin{aligned} D &= (\lambda-1)^2(\lambda+2) \\ D_x &= -(\lambda-1)^2(\lambda+1) \\ D_y &= (\lambda-1)^2 \\ D_z &= (\lambda-1)^2(\lambda+1)^2 \end{aligned}$$

Diskusija:

1. Ako je $D \neq 0$, tj. $\lambda \neq 1$ i $\lambda \neq -2$, tada je sistem saglasan i rješenja su data sa

$$\begin{aligned} x &= \frac{D_x}{D}=\frac{-(\lambda-1)^2(\lambda+1)}{(\lambda-1)^2(\lambda+2)}=-\frac{\lambda+1}{\lambda+2} \\ y &= \frac{D_y}{D}=\frac{(\lambda-1)^2}{(\lambda-1)^2(\lambda+2)}=\frac{1}{\lambda+2} \\ z &= \frac{D_z}{D}=\frac{(\lambda-1)^2(\lambda+1)^2}{(\lambda-1)^2(\lambda+2)}=\frac{(\lambda+1)^2}{\lambda+2}. \end{aligned}$$

2. Ako je $D=0$, tj. $\lambda=1$ ili $\lambda=-2$, tada je sistem neodređen.

* Za $\lambda=1$ imamo da je $D=D_x=D_y=D_z=0$ pa sistem ima beskonačno mnogo rješenja.
Uvrštavajući $\lambda=1$ u početni sistem, dobijamo da je sistem ekvivalentan samo jednoj jednačini $x+y+z=1$. Budući da imamo jednu jednačinu a tri nepoznate, onda dvije nepoznate biramo proizvoljno. Neka su npr. $y=\alpha$ i $z=\beta$, $(\alpha,\beta\in\mathbb{R})$. Tada je $x=1-\alpha-\beta$, pa je rješenje sistema

$$(x,y,z)=(1-\alpha-\beta,\alpha,\beta) \qquad (\alpha,\beta\in\mathbb{R}).$$

** Za $\lambda = -2$ imamo da je $D = 0$ ali je $D_x = 9$ pa sistem nema rješenja.

Zadatak 9.11 *U zavisnosti od realnog parametra a diskutovati rješenja sistema*

$$\left.\begin{array}{r}(a-1)\,x+y-z=2a-1\\(2a-5)\,x+4y-5z=a+2\\(2a-3)\,x+y+(a-3)\,z=a+1\end{array}\right\}.$$

Rješenje:
Izračunajmo determinante D, D_x, D_y, D_z :

$$\begin{aligned}D &= \begin{vmatrix}a-1 & 1 & -1\\2a-5 & 4 & -5\\2a-3 & 1 & a-3\end{vmatrix} = 2a(a-2)\\D_x &= \begin{vmatrix}2a-1 & 1 & -1\\a+2 & 4 & -5\\a+1 & 1 & a-3\end{vmatrix} = (a-2)\,(7a-5)\\D_y &= \begin{vmatrix}a-1 & 2a-1 & -1\\2a-5 & a+2 & -5\\2a-3 & a+1 & a-3\end{vmatrix} = -a\,(a-2)\,(3a-1)\\D_z &= \begin{vmatrix}a-1 & 1 & 2a-1\\2a-5 & 4 & a+2\\2a-3 & 1 & a+1\end{vmatrix} = -\,(a-2)\,(9a-5)\,.\end{aligned}$$

Diskusija:

1. Ako je $D \neq 0$ tj $2a(a-2) \neq 0$ a to je slučaj kada je $a \neq 0$ i $a \neq 2$, tada je sistem saglasan i rješenja su data sa

$$\begin{aligned}x &= \frac{D_x}{D} = \frac{(a-2)\,(7a-5)}{2a(a-2)} = \frac{7a-5}{2a}\\y &= \frac{D_y}{D} = \frac{-a\,(a-2)\,(3a-1)}{2a(a-2)} = -\frac{3a-1}{2}\\z &= \frac{D_z}{D} = \frac{-\,(a-2)\,(9a-5)}{2a(a-2)} = -\frac{9a-5}{2a}.\end{aligned}$$

2. Ako je $D = 0$, tj. $a = 0$ ili $a = 2$, tada je sistem neodređen.

* Za $a = 2$ imamo da je $D = D_x = D_y = D_z = 0$ pa sistem ima beskonačno mnogo rješenja.
Uvrštavajući $a = 2$ u početni sistem, dobijamo da je sistem jednačina

$$\left.\begin{array}{r}x+y-z=3\\-x+4y-5z=4\\x+y-z=3\end{array}\right\}.$$

Ispitajmo saglasnost ovog sistema: Proširena matrica sistema je

$$A_p = \begin{bmatrix}1 & 1 & -1 & 3\\-1 & 4 & -5 & 4\\1 & 1 & -1 & 3\end{bmatrix}.$$

Odredimo njen rang:

$$\begin{aligned}A_p &= \begin{bmatrix}1 & 1 & -1 & 3\\-1 & 4 & -5 & 4\\1 & 1 & -1 & 3\end{bmatrix}\\&\sim \begin{bmatrix}1 & 1 & -1 & 3\\0 & 5 & -6 & 7\\0 & 0 & 0 & 0\end{bmatrix}\begin{array}{r}V_1\ \textit{prepisana}\\V_1+V_2\\V_1-V_2\end{array}\end{aligned}$$

Formirajmo novi sistem

$$\left.\begin{array}{r}x+y-z=3\\5y-6z=7\end{array}\right\}.$$

Ovaj sistem ima tri nepoznate a dvije jednačine, pa jednu nepoznatu biramo proizvoljno. Neka je, recimo $z = \alpha, \alpha \in \mathbb{R}$. Sada, iz druge jednačine imamo

$$5y - 6\alpha = 7 \implies y = \frac{7-6\alpha}{5}.$$

Uvrštavajući $y = \frac{7-6\alpha}{5}$ i $z = \alpha$ u prvu jednačinu dobijamo

$$x + \frac{7-6\alpha}{5} - \alpha = 3 \implies x = \frac{8-\alpha}{5}.$$

Dakle, rješenje sistema je

$$(x, y, z) = \left(\frac{8-\alpha}{5}, \frac{7-6\alpha}{5}, \alpha\right)$$

** Za $a = 0$ imamo da je $D = 0$ ali je $D_x \neq 0$ pa sistem nema rješenja.